FUNCTIONAL ANALYSIS II @ EPFL, SPRING 2025
MOCK EXAM

MATHIAS BRAUN

Abstract. This is the exam from Michele Dolce and Lucio Galeati's preceding course. Try to solve it within 180 minutes using only your allower A4 sheet with handwritten notes (on both sides). This mock exam makes no claim to completeness or comparability with respect to the selection of topics, level of difficulty, or types of questions in relation to the actual exam.

Spoiler alert. The exam starts on the next page. You may not want to read through the problems before you start the exam in order to avoid bias.

Problem 1 (True-or-false questions)**.** For each question, mark (without crossing out) the box TRUE if the statement is always true or the box FALSE if it is not always true (i.e. if it is sometimes false).

(i) Let $B_1(0)$ be the unit ball in $\mathbf{R}^d$. There exists no continuous function $f\colon \overline{B}_1(0) \to \partial B_1(0)$ such that $f(x) = x$ for every $x \in \partial B_1(0)$.

(ii) Let $\Omega \subset \mathbf{R}^d$ be open and bounded and let $f \in L^\infty(\Omega)$ be a function such that $\int_\Omega f(x)\,\varphi(x)\,\mathrm{d}x = 0$ for every $\varphi \in \mathcal{D}(\Omega)$. Then $f = 0$ a.e.

(iii) Let X be a locally convex topological vector space with the Heine–Borel property (each closed bounded set in X is compact). Then X is finite-dimensional.

(iv) Let X be a Hausdorff topological vector space and Y be a finite-dimensional subspace of X. Then Y is closed.

(v) Let X and Y be Banach spaces, $F\colon X \to Y$ be Gâteaux-differentiable at $x \in X$. Then F is Fréchet differentiable at x.

(vi) Let $K \subset \mathbf{R}^d$ be nonempty and compact, $f\colon K \to K$ continuous. Then f has a fixed point in K.

(vii) The Fourier transform $\mathcal{F}$ is a continuous linear operator from $\mathcal{D}(\mathbf{R}^d)$ to itself.

(viii) Let X and Y be Banach spaces, $U \subset X$ open and convex, $F \in C^1(U, Y)$. Then for every $v, w \in U$ there exists $\xi \in U$ such that

$$\|F(v) - F(w)\|_Q \le \|F'(\xi)\|_{\mathcal{L}(X;Y)}\,\|v - w\|_X.$$

Problem 2 (Minkowski functionals)**.** Let (X, τ) be a LCTVS and $A \subset X$ be a given absorbing, balanced, convex set. Recall the Minkowski functional p_A is

$$p_A(x) := \inf\{t > 0 : x \in tA\}.$$

In the following, you may use without proving it that p_A is a seminorm on X.

(i) Show $\{p_A < 1\} \subset A \subset \{p_A \le 1\}$.

(ii) Show that, if $W \subset X$ is open and $x \in W$, then there exists $\delta > 0$ such that $\lambda x \in W$ for all $\lambda \in (1-\delta, 1+\delta)$. Deduce that, if A is open, then $A = \{p_A < 1\}$.

(iii) Show $\{p_A = 1\} \subset \overline{A}$, where $\overline{A}$ denotes the closure of A. You can use without proof the following fact: $x \in \overline{A}$ if and only if, for any open $W \subset X$ such that $x \in W$, it holds $W \cap A \neq \emptyset$.

(iv) Let A be open with respect to τ. Show $p_A\colon X \to \mathbf{R}$ is continuous.
Hint. Verify continuity of the Minkowski functional p_A around any fixed $x \in X$. You might want to look at neighbourhoods of x of the form $x + \varepsilon U$, for a well chosen open set U.

(v) Let A be open with respect to τ. Show $\overline{A} = \{p_A \le 1\}$.
Hint. Start by showing that $\{p_A \le 1\}$ is closed by using (iv). Then combine the previous points.

Problem 3 (Principal value)**.** Recall that the principal value p.v.$(1/x)$ is the element of $\mathcal{D}'(\mathbf{R})$ defined as

$$\text{p.v.}\Big(\frac{1}{x}\Big)(\varphi) := \lim_{\varepsilon \to 0^+} \int_{\mathbf{R}\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(x)}{x}\,\mathrm{d}x$$

(i) Consider the map $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) := \log|x|$ if $x \neq 0$ and $f(0) := 0$. Justify why the assignment $T_f(\varphi) := \int_{\mathbf{R}} f(x)\varphi(x)\,\mathrm{d}x$ is a well-defined as a distribution, i.e. $T_f \in \mathcal{D}'(\mathbf{R})$. Show that $T_f' = \text{p.v.}(1/x)$, where T' denotes the distributional derivative of T.

Hint. You may use without proof that for every $\psi \in \mathcal{D}(\mathbf{R})$,

$$\int_{\mathbf{R}} f(x)\,\psi(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0^+} \int_{\mathbf{R}\setminus(-\varepsilon,\varepsilon)} f(x)\,\psi(x)\,\mathrm{d}x.$$

(ii) Recall that, for $s\mathbf{R}$, a distribution $T \in \mathcal{D}(\mathbb{R})$ is s-homogeneous if for every $\varphi \in \mathcal{D}(\mathbf{R})$ and every $\lambda > 0$,

$$T\big(\lambda^{-1}\,\varphi(\lambda^{-1}\cdot)\big) = \lambda^s\,T(\varphi).$$

Let $T \in \mathcal{D}(\mathbf{R})$ be s-homogeneous. Show T' is $(s-1)$-homogeneous.

(iii) Prove the distribution T_f from (i) is not 0-homogeneous, but its derivative p.v.$(1/x)$ is (-1)-homogeneous.

Problem 4 (Integral equations)**.** Consider the Banach space $X := C([0,1];\mathbf{R})$, equipped with the norm $\|u\|_X := \sup_{x\in[0,1]} |u(x)|$. Define $F\colon \mathbf{R}\times X \to X$ by

$$F(\lambda,u)(x) := u(x) - u^2(x) - \lambda\int_0^1 (1+u(y))^2\,\mathrm{d}y.$$

(i) Find all the solutions to the equation $F(0,u) = 0$ with $u \in X$.

(ii) Show F is C^1 on $\mathbb{R}\times X$. (The product space $Y = \mathbb{R}\times X$ is endowed with the norm $\|(\lambda,u)\|_Y := \max\{|\lambda|, \|u\|_X\}$.)

(iii) Prove there exist $\delta, r \in (0,1/2)$ small enough such that whenever $|\lambda| \le \delta$ there exists a unique solution u to the equation

$$u(x) = u^2(x) + \lambda\int_0^1 (1+u(y))^2\,\mathrm{d}y$$

with the property $\|u\|_X \le r$.

Problem 5 (Integral kernels)**.** Let $f\colon \mathbf{R}^d \to \mathbf{R}^d$ be a continuous, globally bounded function. Let $\kappa \in C^1([0,1]\times[0,1];\mathbf{R})$ be a *kernel*. Prove the following claims.

(i) For every $x \in \mathbf{R}^d$, there exists a solution $y \in C([0,1];\mathbf{R}^d)$ to the following *nonlinear Volterra equation*, where $t \in [0,1]$:

$$y(t) = x + \int_0^t \kappa(t,r)\,f(y(r))\,\mathrm{d}r. \tag{0.1}$$

Hint. Use the Arzelà–Ascoli's theorem to verify precompactness of sets in $C([0,1];\mathbf{R}^d)$.

(ii) Additionally assume f is globally Lipschitz, in the sense that there exists $C_f > 0$ such that $|f(x) - f(y)| \le C_f\,|x-y|$ for all $x, y \in \mathbf{R}^d$. Show there exists $\Delta > 0$ sufficiently small such that any two solutions y_1 and y_2 to (0.1) coincide on $[0,\Delta]$.

(iii) Under the hypothesis of (ii), reiterate the argument therein on intervals $[k\Delta,(k+1)\Delta]$ to conclude y_1 and y_2 coincide on $[0,1]$. Namely, uniqueness for (0.1) holds.

(iv) Show the results from (i) still hold under the following weaker assumption on κ: there exist $\delta \in (0,1]$ and $C > 0$ such that
- $\int_0^t |\kappa(t,r)|^2\,\mathrm{d}r \le C$ for all $t \in [0,1]$;
- $\int_0^s |\kappa(t,r) - \kappa(s,r)|\,\mathrm{d}r \le C|t-s|^\delta$ for all $s, t \in [0,1]$ with $s \le t$.